3.1: Introduction to Second-Order Linear Equations

A second-order linear differential equation is of the form

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \quad \text{or} \quad y'' = f(x, y, y'). \tag{1}$$

(1) can be rearranged as

$$G(x, y, y'y'') = 0 \tag{2}$$

or

$$A(x)y'' + B(x)y' + C(x)y = F(x) \tag{3}$$

or

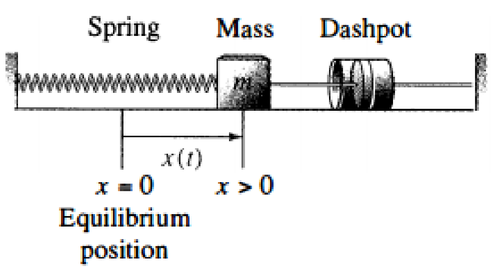
$$y'' + p(x)y' + q(x)y = f(x). \tag{4}$$

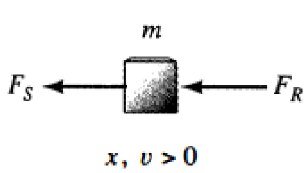
Definition 1.

(a) Anytime that the right-hand side of (3) is identically 0; i.e. $F(x) = 0$, the second-order linear equation is called **homogeneous**.

(b) If the differential equation is not homogeneous it is called **nonhomogeneous**.

A Typical Application. Linear differential equations frequently appear as mathematical models of mechanical systems and electrical circuits.





Using Newton's Second Law of Motion ($F = ma$) along with the diagrams to the left we get the second-order linear differential equation

$$mx'' = F_S + F_R \quad \text{or} \quad mx'' + cx' + kx = F(t). \tag{5}$$

When $F(t) = 0$ in (5) we get a homogeneous equation which describes the free vibrations of the mass. When $F(t) \neq 0$ in (5) we get a nonhomogeneous equation which describes the forced vibrations of the mass under the influence of an external force $F(t)$.

Theorem 1. (Principle of Superposition for Homogeneous Solutions)
Let y_1 and y_2 be two solutions to the homogeneous linear equation (4); i.e. $f(x) = 0$. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution to (4).

Example 1. First check that $y_1 = \cos x$ and $y_2 = \sin x$ are solutions to the differential equation

$$y'' + y = 0.$$

Then check that $y = c_1 y_1 + c_2 y_2$ is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Theorem 2. (Existence and Uniqueness for Linear Equations)
Suppose that p, q and f are continuous on the open interval I in (4) with I containing the point a. Then, given any numbers b_0 and b_1, (4) has a unique solution with the initial conditions

$$y(a) = b_0 \quad y'(a) = b_1.$$

Example 2. We saw in Example 1 that $y = b_0 \cos x + b_1 \sin x$ is a solution to

$$y'' + y = 0.$$

Solve for b_0 and b_1 given that $y(0) = 3$ and $y'(0) = -2$.

Exercise 1. Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions to the differential equation

$$y'' - 2y' + y = 0.$$

Next find the unique solution with the initial conditions $y(0) = 3$ and $y'(0) = 1$.

Definition 1. Two functions defined on an open interval I are said to be **linearly independent** if neither is a constant multiple of the other. If they are not linearly independent, they are said to be **linearly dependent.**

General Solutions. Consider (4) when $p(x)$ and $q(x)$ are constant and $f(x) = 0$. The resulting homogeneous differential equation is given by

$$y'' + py' + qy = 0. \tag{6}$$

Theorem 2 tells us that in this case there are always two linearly independent solutions.

Theorem 3. (Wronskians of Solutions)
Given two solutions to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

the **Wronskian** is defined as

$$W = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - gf'.$$

(a) If f and g are linearly independent, then $W(f, g) \equiv 0$.

(b) If f and g are linearly dependent, then $W(f, g) \neq 0$ at each point.

Theorem 4. (General Solutions of Homogeneous Equations)
If y_1 and y_2 are linearly independent solutions to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

and p and q are continuous on I, then every solution y is of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for some constants c_1 and c_2.

Example 3. Verify that $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ are solutions to

$$y'' - 4y = 0.$$

Next verify that $y_3 = \cosh 2x$ and $y_4 = \sinh 4x$ are also solutions. What does this imply about y_3 and y_4 with respect to y_1 and y_2?

Linear Second-Order Equation with Constant Coefficients.

We return our attention to (6). We replace $y^{(n)}$ with r^n; i.e. $y = 1$, $y' = r$, $y'' = r^2$, etc. to get the **characteristic equation** for (6)

$$r^2 + pr + q = 0. \tag{7}$$

Theorem 5. (Distinct Real Roots)
If the roots r_1 and r_2 of the characteristic equation (7) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution to (6).

Theorem 6. (Repeated Roots)
If the characteristic equation (7) has repeated (necessarily real) roots $r_1 = r_2$, then

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is a general solution to (6).

Example 4. Find the general solution of

$$2y'' - 7y' + 3y = 0.$$

Example 5. Solve the initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = 5, y'(0) = -3.$$

Homework. 1-47 (odd)